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On Einstein's stationary spherically symmetric cluster of particles

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Abstract. It is shown that Einstein equations allow a special class of stationary solutions which correspond to spherically symmetric clusters of particles in circular motions, the total angular momentum of the cluster being zero, and all orbits being performed with the same period. The mass density of such clusters is everywhere regular and positive, decreasing with increasing radius.

1. Introduction

To investigate the mathematical and physical significance of the Schwarzschild singularity, Einstein (1939) in an ingenious way introduced rotation without angular momentum in a system with spherical symmetry. He considered a stationary cluster of particles moving in circular orbits about the centre of symmetry under the influence of the gravitational field produced by all of them together. To have spherical symmetry it was assumed that the phases of motion and the orientation of orbits were perfectly at random. For such a distribution Schwarzschild singularities do not exist in physical reality, because if a cluster of given mass shrinks to the Schwarzschild radius its outermost particles would attain velocities greater than that of light.

The aim of the present work is to investigate a similar stationary spherically symmetric cluster of particles under a constraint of motion. If one assumes that all orbits are performed with the same period, for a given gravitational mass the radius of the cluster depends only on the period, and has a minimum which is three halves the Schwarzschild radius. Further, one can construct solutions corresponding to unbounded clusters. The mass density of such clusters, bounded or not, is everywhere regular and positive, decreasing with increasing radial distance.

2. Field equations

With a suitable choice of spherical coordinates, $x^{\mu} = (x^0, r, \theta, \phi)$, it is possible to obtain stationary spherically symmetric line elements in the form (Anderson 1967, we use his notation and conventions)

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. The Einstein field equations are given by

$$G^{\mu}_{\nu} \equiv R^{\mu}_{\nu} - (R/2)\delta^{\mu}_{\nu} = -(8\pi G/c^4)T^{\mu}_{\nu}, \tag{2}$$

where T^{μ}_{ν} is the energy–stress tensor. For a stationary spherical symmetric cluster of particles one can consider this tensor in the form

$$T^{\mu}_{\nu} = c^2 \rho \,\mathrm{diag}(1+\alpha^2, 0, -\alpha^2/2, -\alpha^2/2), \tag{4}$$

where $\alpha(r)$ is some function related to the components of the velocity vectors of the particles, and $\rho(r)$ is a continuous mass distribution corresponding to the whole of the particles.

Inserting the expressions for $g_{\mu\nu}$ and T^{μ}_{ν} from (1) and (4) respectively into the field equations (2) we obtain

$$G^0_0 \equiv \mathrm{e}^{-\lambda}(r^{-2} - r^{-1}\lambda_1) - r^{-2} = -8\pi G\rho(1+\alpha^2)/c^2, \tag{5}$$

$$G^1_1 \equiv \mathrm{e}^{-\lambda}(r^{-2} + r^{-1}\nu_1) - r^{-2} = 0 \tag{6}$$

and

$$G^2_2 \equiv G^3_3 \equiv \mathrm{e}^{-\lambda}(2r\nu_{11} + r\nu_1^2 + 2\nu_1 - r\nu_1\lambda_1)/4r = 4\pi G\rho\alpha^2/c^2, \tag{7}$$

where the subscript 1 means $\mathrm{d}/\mathrm{d}r$. We can simplify the task of obtaining the solutions of this set of equations. Indeed, contracted Bianchi identities $G^{\mu}_{\nu;\mu} \equiv 0$ connected to the Einstein equations (2) impose the equations of motion $T^{\mu}_{\nu;\mu} = 0$; from these, the equation $\nu = 1$ is the only one which does not vanish identically, and gives

$$r\nu_1 = 2\alpha^2/(1+\alpha^2). \tag{8}$$

The four equations (5) to (8) are not independent, however. We shall conveniently consider (5), (6) and (8) as the equations of our problem. Thus we have three equations to be satisfied by four unknown functions (ν, λ, ρ and α).

Our purpose is to investigate the distribution under a particular constraint of motion, so we choose α arbitrarily. Equation (8) determines ν. Substitution of ν in equation (6) determines λ. One then can easily find the mass density ρ from equation (5).

Let us put $\alpha^2 = \omega^2 r^2/c^2$ where ω is an arbitrary positive constant. Then from equations (5), (6) and (8) we have

$$\rho = 3\omega^2(4\pi G)^{-1}(1+3\omega^2 r^2/c^2)^{-2}, \tag{9}$$

$$\mathrm{e}^{\lambda} = (1+3\omega^2 r^2/c^2)(1+\omega^2 r^2/c^2)^{-1} \tag{10}$$

and

$$\mathrm{e}^{\nu} = A(1+\omega^2 r^2/c^2), \tag{11}$$

where A is a constant of integration.

The constants A and ω will be interpreted from the boundary conditions and geodesic equations.

3. Motion of particles in the gravitational field

The motion of any particles in the field of others is governed by the geodesic equation

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}s^2} + \begin{Bmatrix} \mu \\ \nu\rho \end{Bmatrix} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}s} = 0, \tag{12}$$

with the supplementary condition

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1. \tag{13}$$

Now we define a time-like Killing vector field τ^μ associated to our spherically symmetric stationary field, that is,

$$\tau^2 \equiv g_{\mu\nu}\tau^\mu\tau^\nu > 0 \qquad \text{and} \qquad \tau_{\mu;\nu}+\tau_{\nu;\mu} = 0; \tag{14}$$

with its help we construct the projection tensor

$$\mathscr{B}^\mu_\nu = \delta^\mu_\nu - \tau^\mu\tau_\nu/\tau^2, \tag{15}$$

and the covariant 'normal' velocity (of an object of velocity $u^\mu = dx^\mu/ds$)

$$v^\mu = \mathscr{B}^\mu_\nu u^\nu(\tau^2)^{1/2}(\tau_\rho u^\rho)^{-1}. \tag{16}$$

The norm of this vector corresponds to the norm of the classical velocity of the object,

$$v_c^2 = -c^2 g_{\mu\nu}v^\mu v^\nu. \tag{17}$$

One can easily verify that $\tau^\mu = (1, 0, 0, 0)$ satisfies (14), and that the corresponding $\mathscr{B}^\mu_\nu = \mathrm{diag}(0, 1, 1, 1)$. In order to simplify calculation of v_c we consider a particle of the cluster in equatorial motion: then its $u^\mu = (u^0, 0, 0, u^3)$, where due to the restriction imposed by (13) $(u^0)^2 = A^{-1}[1+(ru^3)^2](1+\omega^2r^2/c^2)^{-1}$. Then the geodesic equation (12) gives for $\mu = 1$ after straightforward calculation $|u^3| = \omega/c$ and thus $u^0 = A^{-1/2}$; finally substituting $u^\mu = (A^{-1/2}, 0, 0, \omega/c)$ into (16) and (17) we get

$$v_c = r\omega(1+\omega^2r^2/c^2)^{-1/2}.$$

Thus we see that for $\omega r \gg c$ we have $v_c \to c$ and for $\omega r \ll c$, $v_c \to r\omega$, so that ω is angular velocity in this limit.

We have shown already that for any equatorial particle $|u^3| \equiv |d\phi/ds| = \omega/c$, irrespective of radial distance from the centre of symmetry. Since we are considering a distribution having spherical symmetry, this result is true for any arbitrary circular orbit. This in turn shows that every particle of the cluster completes a revolution in the same proper time $2\pi/\omega$. However large the cluster may be, the outermost particles have classical velocities that only tend to that of light.

4. Bounded clusters

Outside the cluster, the field is represented by Schwarzschild's solution which is given by

$$ds^2 = [1-2Gm/(c^2r)](dx^0)^2 - [1-2Gm/(c^2r)]^{-1}\,dr^2 - r^2(d\theta^2+\sin^2\theta\,d\phi^2). \tag{18}$$

Since on the boundary $r = a$ one must have continuity of all $g_{\mu\nu}$, we obtain by comparison with (10) and (11)

$$A = (1+3\omega^2a^2/c^2)^{-1} \tag{19}$$

and

$$m = (1+3\omega^2a^2/c^2)^{-1}\omega^2a^3/G. \tag{20}$$

Incidentally we verify that dg_{00}/dr is also continuous on the boundary, but dg_{11}/dr is not.

If one substitutes the expression for m in (18) one obtains for $r \geqslant a$

$$g_{00} = (1+3\omega^2a^2/c^2)^{-1}[1+(3-2a/r)\omega^2a^2/c^2]; \tag{21}$$

this explicity shows that whatever the size of the cluster is, the Schwarzschild singularity does not appear. However, from (20) we get

$$\omega^2 = mGa^{-2}(a-3Gm/c^2)^{-1}, \tag{22}$$

and we see that for a fixed value of m the cluster can be shrunk to any radius greater than

$$a_{\text{min}} = \tfrac{3}{2}r_{\text{S}}, \tag{23}$$

where $r_{\text{S}} = 2Gm/c^2$ is the Schwarzschild radius associated with m. This result coincides with Einstein's general inequality

$$r_{\text{iso}} > Gm(1+\sqrt{3/2})/c^2$$

where the isotropic radial coordinate r_{iso} is related to our r by

$$r = [1+Gm/(2c^2r_{\text{iso}})]^2r_{\text{iso}}.$$

5. Unbounded clusters

For unbounded clusters we take $g_{00} = 1$ at the origin, so $A = 1$ in equation (11).

Since mass density ρ and gravitational potential g_{11} do not depend on A, these are given by the same expressions (9) and (10).

The scalar curvature R, which can easily be computed from (2) and (4), is proportional to the mass density ρ:

$$R = 8\pi G\rho/c^2. \tag{24}$$

6. Conclusions

For bounded clusters we see that near the origin ρ and g_{11} tend to their classical values, the same happening to g_{00} in the case of 'slow' ($\omega a \ll c$) clusters. With increasing distance inside the cluster we have a decreasing ρ and increasing g_{00} and $|g_{11}|$; on the boundary we have the values

$$g_{00} = |g_{11}|^{-1} = (1+\omega^2a^2/c^2)(1+3\omega^2a^2/c^2)^{-1}.$$

Outside the cluster we have $g_{00} = |g_{11}|^{-1}$ tending monotonically to their Minkowski value at infinity.

From equation (21) it is evident that the Schwarzschild singularity does not appear in any region of the bounded cluster in striking contrast to the incompressible fluid sphere of Schwarzschild (1916).

In unbounded clusters we have near the centre ρ, g_{00} and g_{11} tending to their classical values. With increasing distance from the centre, ρ decreases monotonically to zero at infinity, $|g_{11}|$ increases monotonically to the value three at infinity, and g_{00} increases monotonically to ∞ at infinity; however, the scalar curvature R decreases monotonically from $6\omega^2/c^2$ at the origin to zero at infinity. The analogue of classical three-velocity of particles is $r\omega$ near the origin, and increases monotonically to the velocity of light c at infinity. We have obtained similar results for distributions having rotational symmetry with zero net angular momentum (Teixeira and Som 1974).

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